Supplementary Notes for ELEN 4810 Lecture 7
The $\mathcal{Z}$-transform

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Disclaimer: These notes are intended to be an accessible introduction to the subject, with no pretense at completeness. In general, you can find more thorough discussions in Oppenheim's book. Please let me know if you find any typos.

Reading suggestions: Oppenheim and Schafer Sections 3.1-3.6

In this lecture, we discuss the $\mathcal{Z}$-transform, a powerful generalization of the discrete-time Fourier transform. In our study of sampling, as well as some of the application in examples in the homework, we've gotten substantial mileage out of the DTFT. The DTFT provides an excellent tool for studying sampling and reconstruction of continuous time signals, and for understanding the properties of *stable* LTI systems. The reason is that a stable LTI system has absolute summable impulse response ($\sum_n |h[n]| < +\infty$), and so the DTFT $H(e^{j\omega})$ exists, and even satisfies several stronger properties such as continuity.

However, for unstable systems, or systems which are not known to be stable ahead of time, the DTFT is less appropriate – it may not even exist! In this lecture, we study a powerful generalization of the DTFT, known as the $\mathcal{Z}$-transform. The $\mathcal{Z}$-transform allows us to work with many systems which may not be known to be stable ahead of time. We will see that even if the system turns out to be stable, the $\mathcal{Z}$-transform can give additional insight into how our design decisions affect the structure of $H(e^{j\omega})$.

A word of warning: the $\mathcal{Z}$ transform is a powerful tool, but it raises some technical issues that are more subtle than anything else we see in this class. We will try to tackle these clearly and carefully in the lecture; please feel free to ask questions!

1 Definition and Examples

Consider the following function $X(z)$ of a complex variable $z \in \mathbb{C}$:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}. \tag{1.1}$$

The function $X(z)$ is well-defined for any z for which the summation in (1.1) converges. A sufficient condition for $X(z)$ to exist is that the series $x[n]z^{-n}$ is absolute summable, i.e.,

$$\sum_{n=-\infty}^{\infty} |x[n]z^{-n}| < +\infty. \tag{1.2}$$

We make a special notation for set of $z \in \mathbb{C}$ for which (1.2) holds: we call this the *region of convergence* (*ROC*). Formally:

$$\mathrm{ROC}\{x\} = \left\{ z \in \mathbb{C} \;\middle|\; \sum_{n=-\infty}^{\infty} |x[n]z^{-n}| < +\infty \right\}. \tag{1.3}$$

The $\mathcal{Z}$*-transform* of a sequence $x[n]$ is defined as the pair $(X, \mathrm{ROC}\{x\})$, where $\mathrm{ROC}\{x\} \subseteq \mathbb{C}$ and $X : \mathrm{ROC}\{x\} \to \mathbb{C}$. That is to say, *the $\mathcal{Z}$-transform consists of both the function $X(z)$, and the region of convergence*. In notation:

$$x \xleftrightarrow{\mathcal{Z}} X(z),\ \mathrm{ROC}\{x\}. \tag{1.4}$$

A first example. Consider $x[n] = u[n]$. Notice that

$$\sum_{n=-\infty}^{\infty} |u[n]z^{-n}| = \sum_{n=0}^{\infty} (|z|^{-1})^n = \begin{cases} \frac{1}{1-|z|^{-1}} & |z| > 1 \\ +\infty & \text{else} \end{cases} \tag{1.5}$$

Thus, $\mathrm{ROC}\{x\} = \{z \mid |z| > 1\}$. The ROC consists of the portion of the complex plane that lies strictly outside the unit circle. Moreover, for z in the ROC,

$$\sum_{n=-\infty}^{\infty} u[n]z^{-n} = \sum_{n=0}^{\infty} z^{-n} = \frac{1}{1-z^{-1}}, \tag{1.6}$$

and so we have

$$u[n] \xleftrightarrow{\mathcal{Z}} \frac{1}{1-z^{-1}},\ |z| > 1. \tag{1.7}$$

Note I. The ROC is rotationally symmetric. The condition (1.2) for the $\mathrm{ROC}\{x\}$ depends only on the magnitude of z:

$$\sum_{n=-\infty}^{\infty} |x[n]z^{-n}| = \sum_{n=-\infty}^{\infty} |x[n]||z|^{-n}, \tag{1.8}$$

and so if z is in the ROC, every z' satisfying $|z'| = |z|$ is also in the ROC. Put another way: the ROC is invariant under rotation around the origin of the complex plane.

Note II (a technicality): ROC vs. "Region where the summation converges". The ROC as defined in (1.3) is *not* the same as

$$\text{"The region on which the summation } \sum_{n=-\infty}^{\infty} x[n]z^{-n} \text{ converges."} \tag{1.9}$$

Absolute summability is a *sufficient* condition for the summation to converge, but it is not a necessary one.[1] To make matters even more confusing, some sources take (1.9) as the definition of the region of convergence. We (and the text) prefer (1.3), because it produces simpler regions, agrees with common usage in signal processing, and because absolute convergence implies a wealth of additional good properties of $X(z)$ on $\mathrm{ROC}\{x\}$.

Note III (another technicality): does the ROC contain ∞? The ROC as defined in (1.3) is a subset of the complex numbers $\mathbb{C}$. It is typically useful to extend it to the "extended complex numbers" $\bar{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, by asserting that $\infty \in \mathrm{ROC}\{x\}$ if and only if

$$\lim_{|z|\to\infty} \sum_{n=-\infty}^{\infty} |x[n]z^{-n}| < +\infty. \tag{1.15}$$

For example, consider

$$x[n] = \begin{cases} 1 & n = 1, \\ 0 & \text{else.} \end{cases} \tag{1.16}$$

Clearly, $X(z) = \sum_n x[n]z^{-n} = z^{-1}$. Notice that for any $z \neq 0$, $\sum_n |x[n]z^{-n}| = |z|^{-1} < +\infty$. So, the region of convergence is the entire complex plane, except for $z = 0$. Moreover, $\lim_{|z|\to\infty} |z|^{-1} = 0$, and so we say that the region of convergence also contains ∞.

In contrast, now consider what happens if

$$x[n] = \begin{cases} 1 & n = -1, \\ 0 & \text{else.} \end{cases} \tag{1.17}$$

In this case, $X(z) = \sum_n x[n]z^{-n} = z$. Now, the ROC is the entire complex plane. However, $\lim_{|z|\to\infty} |z| = +\infty$, and so ∞ is not in the ROC: the ROC contains all of the extended complex numbers, except for ∞.

Because the summation $\sum_{n=-\infty}^{\infty} z^{-n}$ contains terms of the form z and z^{-1}, we have to take special care with $z = \infty$ and $z = 0$ in determining the ROC.

[1] For example, if we take

$$x[n] = \begin{cases} \frac{1}{n} & n \geq 1 \\ 0 & \text{else} \end{cases} \tag{1.10}$$

it is not difficult to show that

$$\mathrm{ROC}\{x\} = \{z \mid |z| > 1\}. \tag{1.11}$$

However, if $z = -1$,

$$\begin{aligned} \sum_{n=-\infty}^{\infty} x[n]z^{-n} &= \sum_{n=1}^{\infty} \frac{(-1)^n}{n} && (1.12) \\ &= \sum_{n=1,3,5,\ldots} \frac{-1}{n} + \frac{1}{n+1} && (1.13) \\ &= \sum_{n=1,3,5,\ldots} \frac{-1}{n(n+1)}. && (1.14) \end{aligned}$$

This summation converges to a finite value. So, $z = -1$ is in the set defined by (1.9), but is not in $\mathrm{ROC}\{x\}$, as defined in (1.3).

2 Relationship to the DTFT

If the region of convergence contains the unit circle $\left\{e^{j\omega} \mid \omega \in \mathbb{R}\right\}$, then

$$X(z)\big|_{z=e^{j\omega}} = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \tag{2.1}$$

$$= X(e^{j\omega}). \tag{2.2}$$

That is to say, when the ROC contains the unit circle, the DTFT of x exists, and is just equal to the $\mathcal{Z}$-transform, evaluated around the unit circle. From the definition of the ROC, it is easy to see that the ROC contains the unit circle if and only if the sequence x is stable – i.e., it is absolute summable. The $\mathcal{Z}$-transform is that it can also be used to study unstable sequences, or sequences / systems that are not known to be stable ahead of time. We next illustrate this through a few examples.

3 Examples

Example I. For $a \in \mathbb{C}$, take $x[n] = a^n u[n]$. Let us first determine $\mathrm{ROC}\{x\}$. Note that

$$\sum_{n=-\infty}^{\infty} |x[n]z^{-n}| = \sum_{n=0}^{\infty} |az^{-1}|^n \tag{3.1}$$

is finite if and only if $|az^{-1}| < 1$, which occurs iff $|z| > |a|$. So

$$\mathrm{ROC}\{x\} = \{z \in \mathbb{C} \mid |z| > |a|\}. \tag{3.2}$$

In this case, the ROC contains $z = +\infty$. For $z \in \mathrm{ROC}\{x\}$, evaluating the geometric summation $\sum_{n=0}^{\infty}(az^{-1})^n$, we obtain that

$$X(z) = \frac{1}{1 - az^{-1}} = \frac{z}{z - a}. \tag{3.3}$$

Thus,

$$x[n] = a^n u[n] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{z}{z-a},\ |z| > |a|. \tag{3.4}$$

Notice that $\mathrm{ROC}\{x\}$ contains the unit circle if and only if $|a| < 1$, in which case evaluating $X(z)$ around the unit circle reproduces the familiar DTFT relationship

$$x[n] = a^n u[n]\ \ (|a| < 1) \quad \overset{\text{DTFT}}{\longleftrightarrow} \quad \frac{1}{1 - ae^{-j\omega}}. \tag{3.5}$$

When $|a| \geq 1$, the DTFT does not exist, but the $\mathcal{Z}$-transform does exist (within its region of convergence).

Example II. Now let us again take $a \in \mathbb{C}$, and

$$x[n] = \begin{cases} -a^n & n \leq -1 \\ 0 & \text{else} \end{cases} \tag{3.6}$$

$$= -a^n u[-n-1]. \tag{3.7}$$

What is $\mathrm{ROC}\{x\}$? We calculate

$$\begin{aligned}
\sum_{n=-\infty}^{\infty} |x[n]z^{-n}| &= \sum_{n=-\infty}^{-1} |a^n z^{-n}| && (3.8)\\
&= \sum_{n=-\infty}^{-1} |az^{-1}|^n && (3.9)\\
&= \sum_{n=1}^{\infty} |a^{-1}z|^n && (3.10)
\end{aligned}$$

which is finite if and only if $|a^{-1}z| < 1$. Hence $z \in \mathrm{ROC}\{x\}$ if and only if $|z| < |a|$. For $z \in \mathrm{ROC}\{x\}$,

$$\begin{aligned}
X(z) &= -\sum_{n=1}^{\infty} (a^{-1}z)^n && (3.11)\\
&= 1 - \sum_{n=0}^{\infty} (a^{-1}z)^n && (3.12)\\
&= 1 - \frac{1}{1 - a^{-1}z} && (3.13)\\
&= \frac{z}{z-a}. && (3.14)
\end{aligned}$$

Thus,

$$x[n] = -a^n u[-n-1] \quad \stackrel{\mathcal{Z}}{\longleftrightarrow} \quad \frac{z}{z-a},\ |z| < |a|. \tag{3.15}$$

The ROC contains the unit circle if and only if $|a| > 1$, in which case we reproduce another familiar DTFT relationship:

$$x[n] = -a^n u[-n-1]\ , (|a| > 1) \quad \stackrel{\mathrm{DTFT}}{\longleftrightarrow} \quad \frac{1}{1 - ae^{-j\omega}}. \tag{3.16}$$

The important observation from these two examples is that the functional form of the $\mathcal{Z}$ transform $X(z)$ was exactly the same – the only difference was the region of convergence. To fully specify the $\mathcal{Z}$ transform, we need to specify both *$X(z)$ and $\mathrm{ROC}\{x\}$.*

Example III. Let us now take $x[n] = \begin{cases} a^n & 0 \le n < N \\ 0 & \text{else} \end{cases}$, where $N \in \mathbb{Z}$. Then

$$\mathrm{ROC}\{x\} = \{z \mid |z| > 0\}. \tag{3.17}$$

Again, the ROC contains ∞. For $z \in \mathrm{ROC}\{x\}$,

$$X(z) = \sum_{n=0}^{N-1} a^n z^{-n} = \begin{cases} \frac{1-(az^{-1})^N}{1-az^{-1}} & az^{-1} \neq 1 \\ N & az^{-1} = 1. \end{cases} \tag{3.18}$$

We sometimes rewrite the first case as

$$X(z) = \frac{1}{z^{N-1}} \frac{z^N - a^N}{z - a}. \tag{3.19}$$

This is a *rational* function of z. It is worth noting that in the previous two examples, $X(z)$ was a rational function as well.

4 Properties of the ROC

We have seen that the region of convergence

$$\mathrm{ROC}\{x\} = \left\{ z \in \bar{\mathbb{C}} \,\middle|\, \sum_{n=-\infty}^{\infty} |x[n]z^{-n}| < +\infty \right\} \tag{4.1}$$

is crucial in specifying the $\mathcal{Z}$-transform of a signal x. We next study the properties of $\mathrm{ROC}\{x\}$ for general sequences x, without making strong assumptions on the functional form of $X(z)$. In the next section, we will see that for rational $X(z)$, much more can be said.

- **Connectivity.** $\mathrm{ROC}\{x\}$ is *connected*: if $r = |z|$ and $z \in \mathrm{ROC}\{x\}$, and $r' = |z'|$ for $z' \in \mathrm{ROC}\{x\}$, then for any $r'' \in [r, r']$, $r'' \in \mathrm{ROC}\{x\}$.

  Proof. We can quickly check this by calculating:

$$\begin{aligned}
\sum_{n=-\infty}^{\infty} |x[n]r''^{-n}| &= \sum_{n=-\infty}^{0} |x[n]||r''|^{-n} + \sum_{n=1}^{\infty} |x[n]||r''|^{-n} && (4.2)\\
&\leq \sum_{n=-\infty}^{0} |x[n]||r'|^{-n} + \sum_{n=1}^{\infty} |x[n]||r|^{-n} && (4.3)\\
&\leq \sum_{n=-\infty}^{\infty} |x[n]z'^{-n}| + \sum_{n=-\infty}^{\infty} |x[n]z^{-n}| && (4.4)\\
&< +\infty. && (4.5)
\end{aligned}$$

  □

- **Finite duration sequences.** If $x[n]$ has *finite duration* – i.e., $x[n] = 0$ for $n < N_1$ and $x[n] = 0$ for $n > N_2$, for some N_1, N_2, then $\mathrm{ROC}\{x\}$ is the entire complex plane $\mathbb{C}$, with the possible exception of 0 and ∞. *Convince yourself that* $0 \in \mathrm{ROC}\{x\}$ *if and only if* $x[n] = 0$ *for* $n > 0$, *and that* $\infty \in \mathrm{ROC}\{x\}$ *if and only if* $x[n] = 0$ *for* $n < 0$.

- **Right-sided sequences.** If $x[n]$ is *right sided* – i.e., $x[n] = 0$ for $n < N$, for some $N \in \mathbb{Z}$, then $\mathrm{ROC}\{x\}$ extends outward: if $z \in \mathrm{ROC}\{x\}$, then $z' \in \mathrm{ROC}\{x\}$ for any $z' \in \mathbb{C}$ such that $|z'| \geq |z|$.

Proof.

$$\begin{aligned}
\sum_{n=-\infty}^{\infty} |x[n]z'^{-n}| &= \sum_{n=N}^{0} |x[n]||z'|^{-n} + \sum_{n=1}^{\infty} |x[n]||z'|^{-n} &(4.6)\\
&\le \sum_{n=N}^{0} |x[n]||z'|^{-n} + \sum_{n=1}^{\infty} |x[n]||z|^{-n} &(4.7)\\
&< +\infty. &(4.8)
\end{aligned}$$

□

For $z' = \infty$, we have to be a little more careful: we also have to check that $N \ge 0$. In particular, if $x[n]$ is a *causal* sequence, the ROC extends outward; if it is nonempty, it includes ∞.

- **Left-sided sequences.** If $x[n]$ is *left-sided* – i.e., $x[n] = 0$ for $n > N$, for some $N \in \mathbb{Z}$, the $\mathrm{ROC}\,\{x\}$ extends inward: if $z \in \mathrm{ROC}\,\{x\}$, then $z' \in \mathrm{ROC}\,\{x\}$ for any z' such that $0 < |z'| \le |z|$.

Proof.

$$\begin{aligned}
\sum_{n=-\infty}^{\infty} |x[n]z'^{-n}| &= \sum_{n=-\infty}^{0} |x[n]||z'|^{-n} + \sum_{n=1}^{N} |x[n]z'^{-n}| &(4.9)\\
&\le \sum_{n=-\infty}^{0} |x[n]||z|^{-n} + \sum_{n=1}^{N} |x[n]z'^{-n}| &(4.10)\\
&< +\infty. &(4.11)
\end{aligned}$$

□

Here, we have to take special care with $z = 0$. If $N \le 0$, then the ROC extends inward to zero; if $x[n] \neq 0$ for some $n > 0$, then the ROC does not contain $z = 0$.

5 Rational $X(z)$; poles and zeros

In all of our examples up to this point $X(z)$ has been *rational* function on the region of convergence:

$$X(z) = \frac{P(z)}{Q(z)} \tag{5.1}$$

with P and Q polynomials in z. In fact, practically all of the $\mathcal{Z}$ transforms of our interest here are rational – including $\mathcal{Z}$ transforms arising from common sequences, and $\mathcal{Z}$ transforms arising in the solution of difference equations. If $X(z)$ is rational, we can deploy the fundamental theorem of algebra, and factor the numerator $P(z)$ and denominator $Q(z)$, giving

$$X(z) \;=\; \alpha\,\frac{\prod_{i=1}^{d}(z-\zeta_i)}{\prod_{\ell=1}^{d'}(z-\rho_\ell)}, \tag{5.2}$$

where $\alpha \in \mathbb{C}$, $d = \deg(P)$, $d' = \deg(Q)$, the ζ_i are the roots of P, and the ρ_ℓ are the roots of Q. In particular, knowing the roots of P and Q (with multiplicity) tells us the function $X(z)$ up to a single (nonzero) scalar α. The roots $\{\zeta_i\}$ and $\{\rho_\ell\}$ are extremely useful for determining the properties of $X(z)$, and hence of $x[n]$. We give them special names. Before continuing, though, it is worth noting that the polynomials $P(z)$ and $Q(z)$ in (5.1) are not uniquely defined. We can always create another pair of polynomials whose quotient is $X(z)$, by multiplying the numerator and denominator by a common factor – e.g., setting $\tilde{P}(z) = P(z)(z-\beta)$ and $\tilde{Q}(z) = Q(z)(z-\beta)$. Conversely, if P and Q have a common root β, the factor $(z-\beta)$ can be removed from both. *When we talk about the roots of P and Q, we assume that P and Q have no common divisors of degree one or higher – $\{\zeta_i\} \cap \{\rho_\ell\} = \emptyset$.*

The *zeros* of a rational function $X(z)$ are defined as the roots of the numerator $P(z)$ in a rational expression $X(z) = P(z)/Q(z)$ in which $P(z)$ and $Q(z)$ have no common roots. We also say that "$X(z)$ has a zero at ∞" whenever $\lim_{|z|\to\infty} |X(z)| = 0$.

The *poles* of a rational function $X(z)$ are defined as the roots of the denominator $Q(z)$ in a rational expression $X(z) = P(z)/Q(z)$ in which $P(z)$ and $Q(z)$ have no common roots. By convention, we also say that "$X(z)$ has a pole at ∞" whenever $\lim_{|z|\to\infty} |X(z)| = +\infty$.

If the numerator $P(z)$ has a repeated root ζ_i (i.e., $P(z) = \bar{P}(z)(z-\zeta_i)^\ell$ with $\bar{P}$ a polynomial), we say that $P(z)$ has a repeated zero of order ℓ (or multiplicity ℓ) at ζ_i. Similarly, if $Q(z)$ has a repeated root ρ, we say that ρ is a repeated (or multiple) pole.

As z approaches a pole ξ_j, the magnitude $|X(z)|$ approaches $+\infty$. Clearly, for a rational $\mathcal{Z}$-transform $X(z)$, there can be no poles in the region of convergence. Depending on the situation, the zeros ζ_i may lie inside or outside the region of convergence. In introducing the $\mathcal{Z}$-transform, we defined $X(z)$ over $\mathrm{ROC}\{x\}$. When $X(z)$ has rational form $X(z) = P(z)/Q(z)$, we mean that

$$\forall\, z \in \mathrm{ROC}\{x\}, \qquad \sum_{n=-\infty}^{\infty} x[n]z^{-n} \;=\; X(z)\left(= \frac{P(z)}{Q(z)}\right). \tag{5.3}$$

Outside of the region of convergence, there is no guarantee that the summation on the left hand side converges. However, once we know that $X(z)$ is rational, we can use the functional form $X(z) = P(z)/Q(z)$ to think about $X(z)$ as a function of a general complex variable $z \in \mathbb{C}$, defined over $\mathbb{C} \setminus \{\rho_\ell\}$. *When we talk about the poles and zeros of a rational function $X(z)$, we consider $X(z)$ to be defined over the entire complex plane. The poles and zeros (as defined above) are the poles and zeros of this function.*

Note IV (another technicality): we only do this for *rational* $X(z)$. The text briefly defines the poles and zeros of a general $\mathcal{Z}$-transform $X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$ to be the points at which $X(z)$ is infinite and zero, respectively. This definition is not consistent with the definition for rational $X(z)$ given above.[2] In this course, we will be interested almost exclusively in $X(z)$ which are rational. This set of $\mathcal{Z}$-transforms is large enough to allow us to solve linear constant coefficient difference equations, and also includes the $\mathcal{Z}$-transforms of most common sequences. So, we will not dwell on the definition of poles and zeros for general functions $X(z)$. Just be careful: repeating the message of the previous section *The poles and zeros of $X(z)$ are the poles and zeros of the rational function $X(z) = P(z)/Q(z)$.*

[2]If the rational function $X(z)$ has a zero ζ_i outside $\mathrm{ROC}\{x\}$, it can happen that $\sum_n x[n]z^{-n}$ does not converge, or is infinite at ζ_i.

ROC's for rational $X(z)$. For rational $X(z)$, knowing the poles and zeros determines the functional form $X(z)$ up to a scalar multiplication. The location of the poles also puts strong constraints on the possible regions of convergence of $X(z)$. The most obvious constraint is that the ROC cannot contain any poles. In addition, we can strengthen our statements about the ROC for left-sided and right-sided sequences:

- **Finite duration sequences.** The $\mathcal{Z}$-transform of a finite duration sequence is *always* rational. The ROC of a finite duration sequence is the entire complex plane, possibly except for $z = 0$ and $z = +\infty$.

- **Right-sided sequence, rational $X(z)$.** If $x[n] = 0$ for $n < N$ for some $N \in \mathbb{Z}$ (x is right-sided), and $X(z)$ is rational, $\mathrm{ROC}\{x\}$ extends outward from the largest-magnitude finite pole ρ_ℓ, possibly including ∞. If $N \geq 0$, the ROC contains ∞; otherwise it does not.

- **Left-sided sequence, rational $X(z)$.** If $x[n] = 0$ for $n > N$ for some $N \in \mathbb{Z}$ (x is left-sided), and $X(z)$ is rational, $\mathrm{ROC}\{x\}$ extends inward from the smallest-magnitude nonzero pole ξ_i, up to (and possibly including) $z = 0$. If $N \leq 0$, the ROC contains 0; otherwise it does not.

- **Two-sided sequence, rational $X(z)$.** If x is two-sided (there does not exist N such that $x[n] = 0$ for all $n < N$ and there also does not exist N such that $x[n] = 0$ for all $n > N$), and $X(z)$ is rational, then $\mathrm{ROC}\{x\}$ will be a ring in the complex plane, bounded on the interior and exterior by poles, and not containing any poles.

We often draw the poles and zeros of $X(z)$ on the complex plane, with zeros marked by an $\circ$ and poles marked by a $\times$. Repeated poles and zeros are typically marked with multiple $\circ$ symbols and multiple $\times$ symbols, according to multiplicity. We call a drawing of the poles and zeros in the complex plane a *pole-zero diagram*. Given the pole-zero diagram, we know $X(z)$ up to a scalar multiple $\alpha \in \mathbb{C}$. Moreover, together with the properties of the ROC, the pole-zero diagram strongly constrains the sequences $x[n]$ that could correspond to $X(z)$; which one actually does depends on the ROC itself. In the lecture, we illustrate this through a few simple examples.

Example IV. Let us consider another example

$$x[n] = \left(\frac{1}{2}\right)^n u[n] + \left(-\frac{1}{3}\right)^n u[n]. \tag{5.4}$$

Then

$$\begin{aligned} X(z) &= \sum_{n=-\infty}^{\infty} \left[\left(\frac{1}{2}\right)^n u[n] + \left(-\frac{1}{3}\right)^n u[n]\right] z^{-n} && (5.5)\\ &= \sum_{n=0}^{\infty} (1/2)^n z^{-n} + \sum_{n=0}^{\infty} (-1/3)^n z^{-n} && (5.6)\\ &= \frac{1}{1 - \frac{1}{2}z^{-1}} + \frac{1}{1 + \frac{1}{3}z^{-1}} \qquad |z| > \max\left\{\frac{1}{2}, \frac{1}{3}\right\} && (5.7)\\ &= \frac{2z(z - 1/12)}{(z - 1/2)(z + 1/3)} \qquad |z| > \max\left\{\frac{1}{2}, \frac{1}{3}\right\}. && (5.8) \end{aligned}$$

Here, $X(z)$ has zeros at $z = 0$ and $z = 1/12$, and poles at $z = 1/2$ and $z = -1/3$. $x[n]$ is a right-sided sequence; $\mathrm{ROC}\{x\}$ extends outward from the outermost pole (at $1/2$).

Example V. We discuss the possible ROC (and signal properties) associated with two pole zero diagrams. In the first, there are poles at $z = a < b < c$, $b < 1$, and $c > 1$. Here, there are four possible regions of convergence. One corresponds to a right-sided sequence, one corresponds to a left-sided sequence, and the other two correspond to two-sided sequences. Exactly one of these ($\mathrm{ROC}\,\{x\} = \{z \mid b < |z| < c\}$) corresponds to a *stable* sequence; its DTFT exists.

Example VI. In the second example, we consider a pole-zero diagram in which there is a repeated zero at $z = 0$ (of multiplicity two) and poles at $z = 1/2$ and $z = -2$. Here, there are three possible ROC's: (i) $|z| < 1/2$, which corresponds to a left-sided sequence, which is unstable, and whose DTFT does not exist (in ℓ^1 sense). (ii) $1/2 < |z| < 2$, which corresponds to a stable sequence (whose DTFT exists), and which is two-sided. (iii) $|z| > 2$ which corresponds to an unstable sequence, whose DTFT does not exist in a strong sense, which is right-sided.

6 $\mathcal{Z}$-transform properties

Example IV above illustrates two important facts about the $\mathcal{Z}$-transform. The first is that because the definition involves a summation, the $\mathcal{Z}$-transform is *linear* in the sequence $x[n]$. However, the second point is that in manipulating $\mathcal{Z}$-transforms, we need to be very careful to keep track of the region of convergence.

Proposition 6.1 (Linearity)**.** *Suppose that* $x_1[n] \xleftrightarrow{\mathcal{Z}} X_2(z), R_1$ *and* $x_2[n] \xleftrightarrow{\mathcal{Z}} X_2(z), R_2$*, then for any* $\alpha, \beta \in \mathbb{C}$*,*

$$\alpha x_1 + \beta x_2 \xleftrightarrow{\mathcal{Z}} (\alpha X_1 + \beta X_2), R, \tag{6.1}$$

with $R_1 \cap R_2 \subseteq R$*.*

The next property generalizes the time-shifting property of the DTFT:

Proposition 6.2 (Time-shifting)**.** *Suppose that* $x[n] \xleftrightarrow{\mathcal{Z}} X(z), R$*. Then*

$$x[n - n_0] \xleftrightarrow{\mathcal{Z}} z^{-n_0} X(z), R', \tag{6.2}$$

where R' *is identical to* R*, with the possible addition or deletion of* $z = 0$ *and* $z = \infty$*.*

Finally, we provide a generalization of the convolution property of the DTFT. The proof is identical in structure to the proof of the convolution property of the DTFT – we just need to take additional care to track the regions of convergence.

Proposition 6.3 (Convolution)**.** *Suppose that* $x_1[n] \xleftrightarrow{\mathcal{Z}} X_1(z), R_1$*, and* $x_2[n] \xleftrightarrow{\mathcal{Z}} X_2(z), R_2$*. Then*

$$x_1 * x_2 \longleftrightarrow X_1(z)X_2(z),\ R, \tag{6.3}$$

with $R_1 \cap R_2 \subseteq R$*.*

Proof. Let

$$y[n] = \sum_{k=-\infty}^{\infty} x_1[k] x_2[n-k] \tag{6.4}$$

Let us compute the $\mathcal{Z}$-transform.

$$
\begin{aligned}
Y(z) &= \sum_{n=-\infty}^{\infty} z^{-n} \sum_{k=-\infty}^{\infty} x_1[k]x_2[n-k] && (6.5)\\
&= \sum_{k=-\infty}^{\infty} x_1[k]z^{-k} \sum_{n=-\infty}^{\infty} x_2[n-k]z^{-(n-k)} && (6.6)\\
&= X_2(z) \sum_{k=-\infty}^{\infty} x_1[k]z^{-k} \qquad z \in R_2 && (6.7)\\
&= X_2(z)X_1(z), \qquad z \in R_2,\, z \in R_1. && (6.8)
\end{aligned}
$$

This is the desired result. □

7 Inverting the $\mathcal{Z}$-transform

In this section, we describe approaches to computing the inverse $\mathcal{Z}$ transform. The most conceptually straightforward approach is to use the following inversion formula, which asserts that $x[n]$ can be obtained by integrating $X(z)$ over any closed contour C in the ROC:

$$x[n] = \frac{1}{2\pi j} \oint_C X(z) z^{n-1} dz. \tag{7.1}$$

In this formula, C can be any closed contour which (i) lies entirely in the ROC, (ii) encircles 0, and (iii) is oriented in a counterclockwise direction.

This formula looks intimidating. To make it more comfortable, consider the case in which the ROC contains the unit circle $\left\{e^{j\omega} \mid \omega \in \mathbb{R}\right\}$. Taking the particular contour $C = e^{j\omega}\Big|_{-\pi}^{\pi}$, and noting that for $z = e^{j\omega}$, $dz = je^{j\omega}d\omega$, we can express the contour integral as

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega, \tag{7.2}$$

which is just the familiar expression for the inverse DTFT. When the DTFT does not exist (i.e., the unit circle is not in the ROC), we can still invert the $\mathcal{Z}$-transform by computing a similar integral around a different circle which *does* lie in the ROC.

It may seem strange that in the inversion formula (7.1), C can be *any* closed contour that encloses zero and lies inside the ROC. The fact that the particular choice of contour does not matter follows from a remarkable result in complex analysis known as the *Cauchy integral theorem*. Proving this is beyond our scope. However, we will briefly justify (7.1) based on the related *Cauchy integral formula*, which asserts that

$$\frac{1}{2\pi j} \oint_C z^{-k} dz = \begin{cases} 1 & k = 1, \\ 0 & \text{else.} \end{cases} \tag{7.3}$$

Based on this, consider the right hand side of (7.1):

$$\begin{aligned}
\frac{1}{2\pi j}\oint_C X(z)z^{n-1}dz &= \frac{1}{2\pi j}\oint_C \sum_k x[k]z^{n-k-1}dz \\
&= \sum_k x[k]\frac{1}{2\pi j}\oint_C z^{n-k-1}dz \\
&= \sum_k x[k]\delta[n-k] \\
&= x[n]. \qquad (7.4)
\end{aligned}$$

Above, the interchange of integration and summation is justified by the fact that C lies within the ROC, and hence the summation converges absolutely.

Easier ways to deduce the inverse. In practice, contour integration is complicated, and it is desirable to avoid it when possible. The most obvious way around it, called the *"inspection method"* in the text, is to use a table of known $\mathcal{Z}$ transforms in conjunction with the properties of the $\mathcal{Z}$-transform to try to guess the inverse.

A more structured approach is to use partial fraction expansion to write $X(z)$ as a superposition of simpler functions, and then invert each of these simpler functions individually. For our applications, we are almost exclusively interested in $\mathcal{Z}$-transforms with a rational functional form. In this situation, we can use polynomial division and partial fraction expansion to express $\mathcal{Z}$ as a superposition of simpler functions. Since the $\mathcal{Z}$ transform is linear, we can then invert it by summing the inverses of these simpler functions. To make this more concrete, let us assume that

$$\begin{aligned}
X(z) &= \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{\ell=0}^{N} a_\ell z^{-\ell}} \qquad (7.5)\\
&= \frac{b_0\prod_{k=1}^{M}(1-\zeta_k z^{-1})}{a_0\prod_{\ell=1}^{N}(1-\rho_\ell z^{-1})} \qquad (7.6)\\
&= \frac{P(z^{-1})}{Q(z^{-1})}. \qquad (7.7)
\end{aligned}$$

We have chosen here to express $X(z)$ as a quotient of polynomials in z^{-1}, rather than polynomials in z. This is for consistency with the text, which mostly describes rational $\mathcal{Z}$-transforms as polynomials in z^{-1}. However, everything we will say about this situation translates into equivalent statements about polynomials in z, with very minor modifications.

In (7.6), the ζ_k are the zeros of $X(z)$, while the ρ_ℓ are the poles. P is a polynomial of degree M; Q is a polynomial of degree N. We first show how to express $X(z)$ as a sum of simpler terms when $M < N$ and the poles $\rho_1, \ldots, \rho_N$ are distinct:

Proposition 7.1 (Partial fraction expansion, distinct poles)**.** *Suppose that $X(z)$ has the form* (7.6)*, with $M < N$ and $\rho_1, \ldots, \rho_N$ distinct. Then*

$$X(z) = \sum_{\ell=1}^{N} \frac{A_\ell}{1-\rho_\ell z^{-1}}. \qquad (7.8)$$

with

$$A_\ell = \left[(1-\rho_\ell z^{-1})X(z)\right]\Big|_{z=\rho_\ell}. \tag{7.9}$$

Proof. By placing everyone the over a common denominator, we obtain

$$\sum_{\ell=1}^{n} \frac{A_\ell}{1-\rho_\ell z^{-1}} \;=\; \frac{\sum_{i=1}^{N} A_i \prod_{k\neq i}(1-\rho_k z^{-1})}{\prod_{\ell=1}^{n}(1-\rho_\ell z^{-1})}. \tag{7.10}$$

Both $X(z)$ and the above expression consist of polynomials of degree at most $N-1$ divided by the polynomial $\prod_{\ell=1}^{n}(1-\rho_\ell z^{-1})$. It is a basic fact in algebra that if two polynomials of degree at most $N-1$ agree at N distinct points, they are equal everywhere.[3] Our choice of coefficients A_ℓ ensures that the two expressions agree at the N points $\rho_1,\ldots,\rho_\ell$, and hence that $X(z) = \sum_{\ell=1}^{N} \frac{A_\ell}{1-\rho_\ell z^{-1}}$. $\square$

The above proposition gives a very simple way of expanding certain rational functions $X(z)$ as sums of simpler functions, which are then easily inverted. However, it requires that in the expression $X(z) = P(z^{-1})/Q(z^{-1})$, $\deg(P) < \deg(Q)$. If this condition does not hold, we can use polynomial long division to write $X(z)$ as a sum of a polynomial in z^{-1} and a rational function $R(z^{-1})/Q(z^{-1})$ with $\deg(R) < \deg(Q)$:

Proposition 7.2 (Polynomial division). *Suppose $X(z) = \frac{P(z^{-1})}{Q(z^{-1})}$, with P and Q polynomials. Then there is a unique expression*

$$X(z) = D(z^{-1}) + \frac{R(z^{-1})}{Q(z^{-1})}, \tag{7.11}$$

with R, D polynomials and $\deg(R) < \deg(Q)$.

Proof. To show that such a representation exists, we give an algorithm for constructing it. For an arbitrary polynomial $H(z^{-1}) = \sum_{\ell=0}^{d} h_\ell z^{-\ell}$, let $\mathrm{lead}(H) = h_d z^{-d}$ denote the leading term. The algorithm is as follows:

Set $D = 0$, $R = P$.

while $\deg(R) \geq \deg(Q)$,

Set $D = D + \frac{\mathrm{lead}(R)}{\mathrm{lead}(Q)}$.

Set $R = R - \frac{\mathrm{lead}(R)}{\mathrm{lead}(Q)}Q$.

end while

To show that this algorithm produces a representation of the desired form, note that because at each iteration $\deg(R) \geq \deg(Q)$, the term $\mathrm{lead}(R)/\mathrm{lead}(Q)$ is always a polynomial in z^{-1}. Hence, D remains a polynomial in z^{-1}, as does R. Moreover, the degree of R decreases by at least one in each iteration. Hence, after a certain number of iterations, the algorithm terminates with D and R polynomials in z^{-1} and $\deg(R) < \deg(Q)$.

[3] This can be deduced from the fact that the Vandermonde matrix constructed from N distinct points has full rank N.

To show that this representation is unique, note that if $X(z) = D'(z^{-1}) + R'(z^{-1})/Q(z^{-1})$ is another factorization of the same form, then

$$Q(z^{-1})(D - D')(z^{-1}) = (R' - R)(z^{-1}). \tag{7.12}$$

The left hand side is either zero (if $D' = D$) or has degree at least $\deg(Q)$. The right hand side has degree strictly smaller than $\deg(Q)$, and so the only possibility is that $D' = D$, which then implies that $R' = R$. $\square$

The proof of the above proposition gives a very practical way for computing D and R, which you may recognize as polynomial long division. If we are confronted with a function $X(z)$ of the form

$$X(z) = \frac{b_0 \prod_{i=1}^{M}(1 - \zeta_i z^{-1})}{a_0 \prod_{\ell=1}^{N}(1 - \rho_\ell z^{-1})}, \tag{7.13}$$

with $\rho_1, \ldots, \rho_N$ distinct, but $M \geq N$, we can use Proposition 7.2 and Proposition 7.1 to express $X(z)$ as

$$\begin{aligned} X(z) &= D(z^{-1}) + \frac{R(z^{-1})}{\prod_{\ell=1}^{N}(1 - \rho_\ell z^{-1})} && (7.14) \\ &= D(z^{-1}) + \sum_{\ell=1}^{N} \frac{A_\ell}{1 - \rho_\ell z^{-1}}, && (7.15) \end{aligned}$$

which can be inverted piece-by-piece.

The final generalization needed is to consider rational $X(z)$ in which there may be repeated poles. In this case, the form of the partial fraction expansion becomes slightly more complicated:

Proposition 7.3. *Suppose that*

$$X(z) = \frac{P(z^{-1})}{Q(z^{-1})} = \frac{P(z^{-1})}{\prod_{\ell=1}^{U}(1 - \rho_\ell z^{-1})^{d_\ell}}, \tag{7.16}$$

where $\deg(P) < \deg(Q)$, *U is the number of unique (distinct) roots of Q, and $\rho_1, \ldots, \rho_U$ are distinct. Then*

$$X(z) = \sum_{\ell=1}^{U} \sum_{k=1}^{d_\ell} \frac{A_{\ell,k}}{(1 - \rho_\ell z^{-1})^k}, \tag{7.17}$$

with

$$A_{\ell,k} = \frac{1}{(d_\ell - k)!(-\rho_\ell)^{d_\ell - k}} \left[\frac{d^{(d_\ell - k)}}{dw^{d_\ell - k}} \, (1 - \xi_\ell w)^{d_\ell} X(w^{-1}) \right] \Big|_{w = \rho_\ell^{-1}} \tag{7.18}$$

An example. The following example is taken from the text. We consider

$$X(z) = \frac{z^{-2} + 2z^{-1} + 1}{\frac{1}{2}z^{-2} - \frac{3}{2}z^{-1} + 1}, \qquad |z| > 1. \tag{7.19}$$

Factoring the numerator and denominator, we obtain

$$X(z) = \frac{(1+z^{-1})^2}{(1-\frac{1}{2}z^{-1})(1-z^{-1})}. \tag{7.20}$$

Thus, there are no repeated poles. However, the numerator and denominator have the same degree, and so we cannot directly apply partial fraction expansion. Using polynomial long division, we obtain that

$$X(z) = 2 + \frac{5z^{-1}-1}{(1-\frac{1}{2}z^{-1})(1-z^{-1})} \tag{7.21}$$

We then express the second term as

$$\frac{A_1}{1-\frac{1}{2}z^{-1}} + \frac{A_2}{1-z^{-1}} \tag{7.22}$$

where

$$A_1 = \left[\frac{5z^{-1}-1}{1-z^{-1}}\right]\Big|_{z=1/2} = -9 \tag{7.23}$$

$$A_2 = \left[\frac{5z^{-1}-1}{1-\frac{1}{2}z^{-1}}\right]\Big|_{z=1} = 8, \tag{7.24}$$

and so

$$X(z) = 2 - \frac{9}{1-\frac{1}{2}z^{-1}} + \frac{8}{1-z^{-1}}. \tag{7.25}$$

Since

$$2\delta[n] \overset{\mathcal{Z}}{\longleftrightarrow} 2 \tag{7.26}$$

$$\left(\frac{1}{2}\right)^n u[n] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{1-\frac{1}{2}z^{-1}}, \quad |z| > 1/2 \tag{7.27}$$

$$u[n] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{1-z^{-1}}, \quad |z| > 1, \tag{7.28}$$

we conclude that

$$x[n] = 2\delta[n] - 9\left(\frac{1}{2}\right)^n u[n] + 8u[n]. \tag{7.29}$$

Power series expansion. The last trick for avoiding heavy calculation in the inverse $\mathcal{Z}$-transform is to use power series expansion. Namely, we seek to express $X(z)$ as a sum $X(z) = \sum_{n=-\infty}^{\infty} \alpha_n z^{-n}$, and then read of the values $x[n] = \alpha_n$. This approach is perhaps best illustrated through an example.

Let us suppose that $X(z) = \log(1 + az^{-1})$, with ROC $|z| > |a|$. We know that $\log(1+s)$ has a convergent Taylor series for $|s| \leq 1$, namely,

$$\log(1+s) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}s^n}{n}. \tag{7.30}$$

Setting $s = az^{-1}$, we obtain

$$\log(1 + az^{-1}) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} a^n z^{-n}}{n}, \tag{7.31}$$

from which we read off

$$x[n] = \begin{cases} \frac{(-1)^{n+1} a^n}{n} & n \geq 1 \\ 0 & \text{else} \end{cases}. \tag{7.32}$$